A bound on the exponent of the cohomology of BC-bundles

I. J. Leary
Centre de Recerca Matematica,
Institut d'Estudis Catalans,
Apartat 50,
E-08193 Bellaterra.

We give a lower bound for the exponent of certain elements in the integral cohomology of the total spaces of principal BC-bundles for C a finite cyclic group. We are mainly interested in the case when the total space is BG for some discrete group G having a central subgroup isomorphic to C. As applications we give a proof of the theorem of A. Adem and H.-W. Henn that a p-group is elementary abelian if and only if its integral cohomology has exponent p, and we exhibit some infinite groups of finite virtual cohomological dimension whose Tate-Farrell cohomology contains torsion of order greater than the l.c.m. of the orders of their finite subgroups. Our examples include a class of groups having similar properties discovered by Adem and J. Carlson. As a third application, we examine the integral cohomology of a class of p-groups expressible as central extensions with cyclic kernel and quotient abelian of p-rank two. For each such G we determine the minimal n such that almost all (i.e. all but possibly finitely many) of the groups $H^i(BG)$ have exponent dividing p^n. The lemma we use to give an upper bound for the exponents of almost all of the groups $H^i(BG)$ applies to any p-group and may be of independent interest. Here, and throughout the paper, the coefficients for cohomology are to be the integers when not otherwise stated, and we write $\mathbf{Z}_n$ for the integers modulo n. The author gratefully acknowledges that this work was funded by the DGICYT.

Proposition 1. *Let C be a cyclic group of order n, and let E be a principal BC-bundle over a connected space X, classified by $\xi \in H^2(X;C)$ of order m. Then for any $i \geq 0$, any element of $H^{2i}(E)$ restricting to the fibre as a generator for $H^{2i}(BC)$ has order divisible by mn.*

Remark. Note that we do not claim that such elements always exist, nor do we rule out the possibility that they have infinite order.

Proof. In [4] Cartan and Eilenberg computed the ring $H^*(BC;R)$ for any coefficient ring R. Recall that we have the following ring isomorphisms:

$$H^*(BC) \cong \mathbf{Z}[z]/(nz), \qquad H^*(BC;\mathbf{Z}_n) \cong \mathbf{Z}_n[x,y]/(ny,nx,y^2-ex),$$

where $e=0$ if n is odd and $e=n/2$ if n is even, and y has degree 1 while x and z have degree two. The natural map from integral to mod-n cohomology sends z to x, and if we let β stand for the Bockstein for the coefficient sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_n \to 0,$$

then it is easy to see that $\beta(y)=z$, and that therefore $\beta(yx^i)=z^{i+1}$.

Now consider the spectral sequence for the given fibration with coefficients in $\mathbf{Z}_n$. By assumption the fundamental group of X acts trivially on the cohomology of BC, and so

$$E_2^{i,j} \cong H^i(X;\mathbf{Z}_n) \otimes H^j(BC;\mathbf{Z}_n).$$

Now $1\otimes yx^j$ represents a generator for $E_2^{0,2j+1}$ and $1\otimes x^j$ represents a generator for $E_2^{0,2j}$. Comparing this spectral sequence with the spectral sequence for the path-loop fibration over an Eilenberg-MacLane space $K(C,2)$ it is easy to see that $d_2(1\otimes y) = \xi$ and $d_2(1\otimes x) = 0$. (In fact, $d_3(1\otimes x) = \xi' \otimes 1$, where ξ' is the image of $\beta(\xi)$ under the map from $H^3(X)$ to $H^3(X;\mathbf{Z}_n)$, and d_4 may be described using the argument given in [8], but we do not need this here.) Now $d_2(1\otimes x^j y) = \xi\otimes x^j$ and $d_2(1\otimes x^j) = 0$, from which it follows that $E_3^{0,2j}$ is generated by $1\otimes x^j$ and $E_3^{0,2j+1}$ by $m(1\otimes yx^j)$. The map from $H^*(E;\mathbf{Z}_n)$ to $H^*(BC;\mathbf{Z}_n)$ factors through $E_\infty^{0,*}$, which is a subgroup of $E_3^{0,*}$, and so we see that the image of $H^{2j+1}(E;\mathbf{Z}_n)$ in $H^{2j+1}(BC;\mathbf{Z}_n)$ must be contained in the subgroup generated by myx^j.

Now recall that the image of the Bockstein β defined above is exactly the elements of integral cohomology of order dividing n. Let $f : BC \to E$ be the inclusion of the fibre of the above fibration. Now let χ be an element of $H^*(E)$ such that $f^*(\chi) = z^{j+1}$ for some j. If χ has infinite order then there is nothing to prove. Otherwise, the order of χ must be a multiple of n (the order of z^{j+1}), say $m'n$, and it remains to show that m divides m'. Now $m'\chi$ has order n, so there exists $\chi' \in H^{2j+1}(E;\mathbf{Z}_n)$ such that $\beta(\chi') = m'\chi$. However, the spectral sequence argument shows that $f^*(\chi')$ is in the subgroup of $H^{2j+1}(BC;\mathbf{Z}_n)$ generated by myx^j and hence $\beta f^*(\chi')$ is in the subgroup of $H^{2j+2}(BC)$ generated by mz^{j+1}, but $\beta f^*(\chi') = f^*\beta(\chi') = f^*(\chi) = m'z^{j+1}$. □

Corollary 1. *Let C be a cyclic subgroup of order n of a group G. If there exists an element of $H^*(BG)$ of order n whose image in $H^*(BC)$ is a generator for $H^{2i}(BC)$ for some i, then C is a direct factor of its centraliser in G.*

Proof. This is just Proposition 1 applied to the principal BC-bundle with total space the classifying space of the centraliser of C. □

Corollary 2. *Let G be a discrete group expressible as a central extension with kernel C cyclic of order n. Let Q be the quotient G/C, and let the extension class of G in $H^2(BQ;C)$ have order m. If G has a normal subgroup N of finite index whose intersection with C is trivial (for example, if G is finite or residually finite), then for infinitely many i, $H^{2i}(BG)$ contains elements of order mn.*

Remark. The condition that the extension class of G has order m may be rephrased as follows: If D is the smallest subgroup of C such that G/D is isomorphic to $(C/D)\times Q$, then D has order m.

Proof. Let G' be the quotient G/N, and let C' be the image of C in G'. Then C' is isomorphic to C and G' is finite. By either Evens' argument using the Norm map from $H^*(BC)$ to $H^*(BG)$ [5,6] or Venkov's argument using Chern classes of a representation of G' restricting faithfully to C' [10], we see that for infinitely many i there exists $\chi' \in H^{2i}(BG')$ whose image in $H^{2i}(BC')$ is a generator. If χ is the image of χ' in $H^*(BG)$, then χ has finite order (dividing the order of G') and its image in $H^{2i}(BC)$ is a generator. Hence by Proposition 1, some multiple of χ has order exactly mn. $\square$

The first example of a group whose Tate-Farrell cohomology contains elements of order greater than the l.c.m. of the orders of its finite subgroups is due to Adem [2]. The following application of Corollary 2 is more closely related to some other examples due to Adem and Carlson [3]. In particular, Corollary 3 may be compared with Theorem 3.1 of [3], which gives stronger cohomological information about a smaller class of groups.

Corollary 3. *With notation and hypotheses as in Corollary 2, assume also that Q has finite cohomological dimension (or equivalently, assume that there is a finite-dimensional CW-complex BQ). Then*

a) G has finite virtual cohomological dimension and hence the Tate-Farrell cohomology groups $\hat{H}^i(G)$ are defined,

b) C consists of all the elements of G of finite order, and

c) $\hat{H}^i(G)$ contains elements of order mn for infinitely many i.

Proof. The subgroup N of G has finite index and is isomorphic to a subgroup of Q, so has cohomological dimension less than or equal to that of Q. Hence G has finite vcd. The group Q is torsion-free, and so any element of $G - C$ has infinite order because its image in Q does. If i is greater than vcd$G$ then $\hat{H}^i(G)$ is isomorphic to $H^i(BG)$, and so the third claim follows from Corollary 2. $\square$

The following Corollary is due to Adem [1] and Henn [7].

Corollary 4. *Let G be a finite p-group. Then G is not elementary abelian if and only if $H^i(BG)$ contains elements of order p^2 for some i if and only if $H^i(BG)$ contains elements of order p^2 for infinitely many i.*

Proof. If G is elementary abelian (i.e. is isomorphic to a product of cyclic groups of order p) then $H^i(G)$ has exponent p for $i > 0$ by the Künneth theorem. Conversely, if G is not elementary abelian then G contains a central subgroup of order p which is not a direct factor, or equivalently, C of order p such that the extension class of G in $H^2(BG/C; C)$ has order p. The result now follows by applying Corollary 2. $\square$

The following application of Proposition 1 is new.

Proposition 2. *For positive integers α, β, γ, δ satisfying the inequalities $0 \leq \gamma - \delta \leq \min\{\alpha, \beta\}$, let $G = G(\alpha, \beta, \gamma, \delta)$ be a p-group with the following*

presentation.

$$G = \langle a, b, c \mid [a,c] = [b,c] = 1 = a^{p^\alpha} = b^{p^\beta} = c^{p^\gamma}, \quad [a,b] = c^{p^\delta} \rangle$$

Now let ϵ be $\max\{\alpha, \beta, 2\gamma - \delta\}$. Then for infinitely many i, $H^i(BG)$ has exponent p^ϵ, and at most finitely many of the groups $H^i(BG)$ have higher exponent.

Remark. It is easy to see that any group having a presentation of the above form for arbitrary $(\alpha, \beta, \gamma, \delta)$ also has a presentation of the above form in which the inequalities are satisfied: If γ is less than δ, then $c^{p^\delta} = c^{p^\gamma} = 1$, and so in this case $G(\alpha, \beta, \gamma, \delta)$ is isomorphic to $G(\alpha, \beta, \gamma, \gamma)$. On the other hand, the order of $[a,b] = c^{p^d}$ is bounded by the orders of a and b given that c is central, and so the order of c is bounded by $p^{\alpha+\delta}$ and $p^{\beta+\delta}$. Thus given a presentation as above but not satisfying the second inequality we could replace γ by $\gamma' = \min\{\alpha+\delta, \beta+\delta\}$ and obtain another presentation of the same group.

Proof. First we recall that for any G and any split surjection from G onto Q, $H^*(BQ)$ occurs as a direct summand of $H^*(BG)$. Now the above group G may be expressed as a split extension with kernel $\langle a, c \rangle$ and quotient $\langle b \rangle \cong \mathbf{Z}/p^\beta$, or as a split extension with kernel $\langle b, c \rangle$ and quotient $\langle a \rangle \cong \mathbf{Z}/p^\alpha$. Hence we deduce that $H^{2i}(BG)$ has elements of exponents p^α and p^β for all $i > 0$.

G may also be viewed as a central extension with kernel $\langle c \rangle$ which is isomorphic to $\mathbf{Z}/p^\gamma$, and quotient isomorphic to $\mathbf{Z}/p^\alpha \oplus \mathbf{Z}/p^\beta$ generated by the images of a and b. The extension class of this extension is easily seen to have order $p^{\gamma-\delta}$, and so it follows from Corollary 1 that for infinitely many i, $H^{2i}(BG)$ contains elements of order $p^{2\gamma-\delta}$.

For the partial converse, note that G has subgroups $\langle a, c \rangle$, $\langle b, c \rangle$, and $\langle a, b^{p^{\gamma-\delta}} \rangle$ of index p^α, p^β and $p^{2\gamma-\delta}$ respectively whose intersection is trivial, and then apply the following Lemma.

Lemma 1. *Let G be a (finite) p-group, let $H_1, \ldots, H_k$ be a family of subgroups of G such that the index $|G : H_j|$ of each H_j is less than or equal to p^n, and suppose that the intersection*

$$\bigcap_{g \in G, 1 \leq j \leq k} H_j^g$$

of the conjugates of the subgroups H_j is trivial. Then $H^i(BG)$ has exponent dividing p^n for all but finitely many i.

Proof. Let Σ_m be the symmetric group on m symbols and let G_n be the Sylow p-subgroup of Σ_{p^n}. Since the index of $(\Sigma_m)^p$ in Σ_{mp} divides exactly once by p an easy induction argument using the transfer shows that for all $i > 0$ and all n, $H^i(BG_n)$ has exponent dividing p^n. If H is a subgroup of G, then the kernel of the permutation representation of G on the cosets of H is the intersection of the conjugates of H. Hence if G has subgroups $H_1, \ldots, H_k$ as in the statement

then G occurs as a subgroup of a product of k symmetric groups on at most p^n symbols, and hence as a subgroup of $(G_n)^k$. The result now follows from the observation due to Adem [1] that for any group G' and any subgroup G, the finite generation of $H^*(BG)$ as an $H^*(BG')$-module implies that at most finitely many of the groups $H^i(BG)$ can have higher exponent than the reduced cohomology $\tilde{H}^*(BG')$. □ □

Remark. The bound given by Lemma 1 for the exponent of almost all of the integral cohomology groups of a p-group is attained for many groups. For example, Proposition 2 shows that the bound is attained for the groups $G(\alpha, \beta, \gamma, \delta)$. We were tempted to conjecture that the bound is always attained, but have recently found a group of order 128 whose index four subgroups intersect non-trivially and whose integral cohomology has exponent four [9]. Adem has conjectured that for G a finite group, if $H^i(BG)$ contains elements of order p^n for some i, then it does so for infinitely many i [1], and Henn has asked if this is the case [7]. We do not know if this holds for the groups $G(\alpha, \beta, \gamma, \delta)$.

References.

[1] A. Adem, Cohomological exponents of **Z**G-lattices, J. Pure and Appl. Alg. **58** (1989), 1–5.

[2] A. Adem, On the exponent of the cohomology of discrete groups, Bull. London Math. Soc. **21** (1989), 585–590.

[3] A. Adem and J. F. Carlson, Discrete groups with large exponents in cohomology, J. Pure and Appl. Alg. **66** (1990), 111–120.

[4] H. Cartan and S. Eilenberg, Homological Algebra, Princeton Univ. Press (1956).

[5] L. Evens, The cohomology ring of a finite group, Trans. Amer. Math. Soc. **101** (1961), 224–239.

[6] L. Evens, A generalization of the transfer map in the cohomology of groups, Trans. Amer. Math. Soc. **108** (1963), 54–65.

[7] H.-W. Henn, Classifying spaces with injective mod-p cohomology, Comment. Math. Helvetici **64** (1989), 200–206.

[8] I. J. Leary, A differential in the Lyndon-Hochschild-Serre spectral sequence, J. Pure and Appl. Alg. **88** (1993), 155–168.

[9] I. J. Leary, Integral cohomology of some wreath products, in preparation.

[10] B. B. Venkov, Cohomology algebras for some classifying spaces, Dokl. Akad. Nauk SSSR **127** (1959), 943–944 (in Russian).